Exam Calculus 2

4 April 2014, 14:00-17:00

university of groningen

The exam consists of 6 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**8+4+3 Points**] Consider the function
$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
   (a) Use the definition of partial derivatives to find $f_x(0,0)$ and $f_y(0,0)$.
   (b) Let a be a nonzero constant. By the substitutions $x(t) = t$ and $y(t) = at$, the function $f(x,y)$ becomes a function of t. Show that this function is differentiable, and find its derivative at $t = 0$.
   (c) Compute $f_x(0,0)\, x'(0) + f_y(0,0)\, y'(0)$. Use this result and the result in part (b) to comment on the applicability of the chain rule.

2. [**10+5 Points**] Let C be the curve parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$,
$$\mathbf{r}(t) = \mathrm{e}^t \cos t\, \mathbf{i} + \mathrm{e}^t \sin t\, \mathbf{j} + \mathrm{e}^t\, \mathbf{k}.$$
   (a) Find the parametrization of C by arc length.
   (b) For each point on C, compute the curvature of C at this point.

3. [**7+8 Points**] Consider the ellipsoid
$$3x^2 + 2y^2 + z^2 = 6.$$
   (a) Compute the tangent plane of the ellipsoid at the point $(x,y,z) = (1,1,1)$.
   (b) Use the Method of Lagrange Multipliers to find the radius of the largest sphere centered at the origin that can fit inside of the ellipsoid.

4. [**11+4 Points**] Consider the vector field
$$\mathbf{F} = x^2\, \mathbf{i} + \cos y \sin z\, \mathbf{j} + \sin y \cos z\, \mathbf{k}.$$
   (a) Show that $\mathbf{F}$ is conservative and find a potential function f for $\mathbf{F}$.
   (b) Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the curve parametrized by $\mathbf{r} : [0,1] \to \mathbb{R}^3$, $\mathbf{r}(t) = (t^2+1)\mathbf{i} + \mathrm{e}^t\mathbf{k} + \mathrm{e}^{2t}\mathbf{k}$ where C is oriented by the tangent vector associated with the parametrization.

5. [**20 Points**] Verify Stokes' Theorem for the flux of the vector field $\mathbf{F} = -2yz\, \mathbf{i} + y\, \mathbf{j} + 3x\, \mathbf{k}$ upward through the part of the paraboloid $z = 5 - x^2 - y^2$ that has $z \geq 1$.

6. [**10 Points**] Let D be a solid region in $\mathbb{R}^3$ with boundary ∂D. Prove that for $\mathbf{r} = x\, \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}$ and $r = |\mathbf{r}|$,
$$\iiint_D \frac{1}{r^2} \mathrm{d}V = \iint_{\partial D} \frac{1}{r^2} \mathbf{r} \cdot \mathrm{d}\mathbf{S}.$$

Solutions

1. (a) By definition

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h-0} = \lim_{h\to 0} \frac{\frac{h^2 0}{h^2+0^2} - 0}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0,$$

and similarly

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h-0} = \lim_{h\to 0} \frac{\frac{0^2 h}{0^2+h^2} - 0}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0.$$

(b) Let $t \in \mathbb{R}$ and $g(t) = f(x(t), y(t))$. Then

$$\begin{aligned} g(t) &= \begin{cases} \frac{(x(t))^2 y(t)}{(x(t))^2+(y(t))^2} & \text{if } (x(t),y(t)) \neq (0,0) \\ 0 & \text{if } (x(t),y(t)) = (0,0) \end{cases} = \begin{cases} \frac{t^2(at)}{t^2+(at)^2} & \text{if } (t,at) \neq (0,0) \\ 0 & \text{if } (t,at) = (0,0) \end{cases} \\ &= \begin{cases} \frac{t^2(at)}{t^2+(at)^2} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} = \begin{cases} \frac{a}{1+a^2} t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} = \frac{a}{1+a^2} t\,. \end{aligned}$$

Hence g is differentiable on $\mathbb{R}$ and

$$g'(t) = \frac{a}{1+a^2}\,.$$

(c) We have $x'(0) = 1$ and $y'(0) = a$. Hence by the results in part (a)

$$f_x(0,0)x'(0) + f_y(0,0)y'(0) = 0 \cdot 1 + 0 \cdot a = 0.$$

This differs from $g'(0)$ for $a \neq 0$. The chain hence does not apply for $a \neq 0$. We conclude that f is not differentiable at $(0,0)$.

2. (a) The arc length is defined as

$$s(t) = \int_0^t |\mathbf{r}'(\tau)|\,\mathrm{d}\tau\,.$$

We have $\mathbf{r}'(t) = \mathrm{e}^t(\cos t - \sin t)\,\mathbf{i} + \mathrm{e}^t(\sin t + \cos t)\,\mathbf{j} + \mathrm{e}^t\,\mathbf{k}$, and hence

$$|\mathbf{r}'(t)| = \mathrm{e}^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1} = \mathrm{e}^t\sqrt{2\cos^2 t + 2\sin^2 t + 1} = \mathrm{e}^t\sqrt{3}.$$

Hence

$$s(t) = \int_0^t \mathrm{e}^\tau \sqrt{3}\,\mathrm{d}\tau = (\mathrm{e}^t - 1)\sqrt{3}.$$

Solving for t gives

$$t(s) = \ln(\sqrt{3}\,s + 1).$$

So the parametrization of C by arc length is given by

$$\begin{aligned} \tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) &= \mathrm{e}^{t(s)}\cos t(s)\,\mathbf{i} + \mathrm{e}^{t(s)}\sin t(s)\,\mathbf{j} + \mathrm{e}^{t(s)}\,\mathbf{k} \\ &= (\sqrt{3}\,s + 1)(\cos\ln(\sqrt{3}\,s+1)\,\mathbf{i} + \sin\ln(\sqrt{3}\,s+1)\,\mathbf{j} + \mathbf{k}) \end{aligned}$$

where $0 \leq s \leq (\mathrm{e}^{2\pi} - 1)\sqrt{3}$.

(b) The curvature κ is defined as

$$\kappa = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|,$$

where $\mathbf{T}$ is the unit tangent vector. By the chain rule

$$\kappa = \frac{1}{|\mathbf{r}'(t)|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right|,$$

From part (a) we get

$$\mathbf{T} = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{1}{\sqrt{3}}\big((\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j} + \mathbf{k}\big)$$

which gives

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} = \frac{1}{\sqrt{3}}\big((-\sin t - \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j}\big)$$

and hence by a similar computation as in part (a)

$$\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right| = \left|\frac{1}{\sqrt{3}}\big((-\sin t - \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j}\big)\right| = \frac{\sqrt{2}}{\sqrt{3}}.$$

The curvature of C at $\mathbf{r}(t)$ is thus

$$\kappa = \frac{1}{\mathrm{e}^t\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{3}\mathrm{e}^{-t}.$$

3. (a) *First method:* The 'upper' half of the ellipsoid can be considered to be the graph of the function

$$f(x,y) = \sqrt{6 - (3x^2 + 2y^2)}.$$

We can thus compute tangent plane of the ellipsoid at $(x,y,z) = (1,1,1)$ from the linearization of f at $(x,y) = (1,1)$ which is given by

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1).$$

Using $f_x(x,y) = -3x/\sqrt{6-3x^2-2y^2}$ and $f_y(x,y) = -2y/\sqrt{6-3x^2-2y^2}$ and hence $f_x(1,1) = -3$ and $f_y(1,1) = -2$ we find for the tangent plane

$$z = L(x,y) = 1 - 3(x-1) - 2(y-1) = 6 - 3x - 2y.$$

Second method: We can view the ellipsoid to be given by a level set of the function

$$F(x,y,z) = 3x^2 + 2y^2 + z^2,$$

and we can hence write the tangent plane as

$$\nabla F(1,1,1) \cdot (x-1, y-1, z-1) = 0.$$

We have $\nabla F(x,y,z) = (6x, 4y, 2z)$ and hence $\nabla F(1,1,1) = (6,4,2)$. For the tangent plane we thus find

$$(6,4,2) \cdot (x-1, y-1, z-1) = 0,$$

or equivalently,

$$z = 6 - 3x - 2y$$

which agrees with the result obtained from the first method.

(b) Let

$$g(x,y,z) = x^2 + y^2 + z^2$$

which gives the square distance of (x,y,z) to the origin. To find the largest sphere fitting inside of the ellipsoid we study the extrema of g under the constraint $F(x,y,z) = 3x^2 + 2y^2 + z^2 = 6$. By the method of Langrange multipliers there is a $\lambda \in \mathbb{R}$ such that $\nabla g = \lambda \nabla F$ at the extremum. This yields the set of equations

$$\begin{aligned} g_x &= \lambda F_x, \\ g_y &= \lambda F_y, \\ g_z &= \lambda F_z, \\ F(x,y,z) &= 6, \end{aligned}$$

i.e.

$$\begin{aligned} 2x &= \lambda 6x, \\ 2y &= \lambda 4y, \\ 2z &= \lambda 2z, \\ 3x^2 + 2y^2 + z^2 &= 6, \end{aligned}$$

which is equivalent to

$$\begin{aligned} x = 0 \cup \lambda &= \tfrac{1}{3}, \\ y = 0 \cup \lambda &= \tfrac{1}{2}, \\ z = 0 \cup \lambda &= 1, \\ 3x^2 + 2y^2 + z^2 &= 6. \end{aligned}$$

This in turn is equivalent to

$$x = y = 0 \cap \lambda = 1 \text{ or } x = z = 0 \cap \lambda = \tfrac{1}{2} \text{ or } y = z = 0 \cap \lambda = \tfrac{1}{3},$$
$$3x^2 + 2y^2 + z^2 = 6$$

or

$$\begin{aligned} x = y = 0, &\quad z = \pm\sqrt{6}, \quad \lambda = 1 \text{ or} \\ x = z = 0, &\quad y = \pm\sqrt{3}, \quad \lambda = \tfrac{1}{2} \text{ or} \\ y = z = 0, &\quad x = \pm\sqrt{2}, \quad \lambda = \tfrac{1}{3}. \end{aligned}$$

We have $g(0,0,\pm\sqrt{6}) = 6$, $g(0,\pm\sqrt{3},0) = 3$ and $g(\pm\sqrt{2},0) = 2$. The greatest sphere that fits inside of the ellipsoid has radius where g has a minimum on the ellipsoid, i.e. the sphere has radius $\sqrt{2}$ which agrees with the smallest semi axis of the ellipsoid.

4. (a) Since $\mathbf{F}$ is defined on a simply connected domain it is sufficient to show that $\nabla \times \mathbf{F} = 0$ in order to prove that $\mathbf{F}$ is conservative.

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & \cos y \sin z & \sin y \cos z \end{vmatrix} \\ &= \Big(\frac{\partial \sin y \cos z}{\partial y} - \frac{\partial \cos y \sin z}{\partial z}\Big)\,\mathbf{i} - \Big(\frac{\partial \sin y \cos z}{\partial x} - \frac{\partial x^2}{\partial z}\Big)\,\mathbf{j} + \Big(\frac{\partial \cos y \sin z}{\partial x} - \frac{\partial x^2}{\partial y}\Big)\,\mathbf{k} \\ &= (\cos y \cos z - \cos y \cos z)\,\mathbf{i} - 0\,\mathbf{j} + 0\,\mathbf{k} = 0. \end{aligned}$$

Let f be a potential function for $\mathbf{F}$, i.e.

$$\frac{\partial f}{\partial x} = x^2 \ (*),$$
$$\frac{\partial f}{\partial y} = \cos y \sin z \ (**),$$
$$\frac{\partial f}{\partial z} = \sin y \cos z \ (***).$$

Integrating both sides of (*) with respect to x gives

$$f(x, y, z) = \frac{1}{3}x^3 + g(y, z),$$

where $g(y, z)$ is an integration constant. Combining this with (**) gives

$$\frac{\partial f}{\partial y} = \cos y \sin z = \frac{\partial g}{\partial y}.$$

Integrating both hand sides of the latter equation with respect to y gives

$$g(y, z) = \sin y \sin z + h(z),$$

where $h(z)$ is an integration constant. Combining with (***) gives

$$\frac{\partial f}{\partial z} = \sin y \cos z = \sin y \cos z + h'(z).$$

Integrating both hand sides of the latter equation with respect to z gives

$$h(z) = c,$$

where $c \in \mathbb{R}$ is an integration constant. Thus

$$f(x, y, z) = \frac{1}{3}x^3 + \sin y \sin z + c.$$

One easily checks that indeed $\nabla f = \mathbf{F}$.

(b) Using that $\mathbf{F}$ is conservative and that f is a potential function for $\mathbf{F}$ we get by the fundamental theorem of line integrals

$$\int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = f(\mathbf{r}(1)) - f(\mathbf{r}(0)).$$

Using $\mathbf{r}(1) = 2\,\mathbf{i} + \mathrm{e}\,\mathbf{j} + \mathrm{e}^2\,\mathbf{k}$ and $\mathbf{r}(0) = 1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k}$ we get

$$\int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \frac{1}{3}2^3 + \sin \mathrm{e} \sin \mathrm{e}^2 - \frac{1}{3} - \sin 1 \sin 1 = \frac{7}{3} + \sin \mathrm{e} \sin \mathrm{e}^2 - \sin^2 1.$$

5. For the present case, Stokes' Theorem should yield

$$\iint_D \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where D is the part of the paraboloid $z = 5 - x^2 - y^2$ which has $z \geq 1$ and ∂D is the boundary of D. As we need to compute the upward flux we choose an orientation on D which is given by a vector pointing upward, and the boundary ∂D is oriented consistently.

We start by computing the right hand side of Stokes' Theorem. We note that the boundary of D is given by

$$\partial D = \{(x, y, z) \in \mathbb{R}^3 \,:\, z = 1,\, x^2 + y^2 = 4\},$$

i.e. ∂D is a circle of radius 2 in the plane $z = 1$ centered at the z-axis. We parametrize this curve using

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$$

with $t \in [0, 2\pi]$. The orientation of ∂D induced by the parametrization $\mathbf{r}$ is such that ∂D is traversed in the counterclockwise direction when looking at ∂D from above. We then have

$$\begin{aligned}\int_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^{2\pi} (-4\sin t, 2\sin t, 6\cos t) \cdot (-2\sin t, 2\cos t, 0)\,\mathrm{d}t \\ &= \int_0^{2\pi} 8\sin^2 t + 4\sin t\cos t\,\mathrm{d}t = 4t - 4\cos t\sin t - 2\cos^2 t\Big|_{t=0}^{t=2\pi} = 8\pi.\end{aligned}$$

For the left hand side of Stokes' Theorem, we first compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -2yz & y & 3x \end{vmatrix} = -(3 + 2y)\,\mathbf{j} + 2z\,\mathbf{k}.$$

For the parametrization of D, we choose $\mathbf{r} : (x, y) \mapsto (x, y, 5 - x^2 - y^2)$ with $x^2 + y^2 \leq 4$. Then $\partial\mathbf{r}/\partial x = (1, 0, -2x)$, $\partial\mathbf{r}/\partial y = (0, 1, -2y)$, and

$$\frac{\partial\mathbf{r}}{\partial x} \times \frac{\partial\mathbf{r}}{\partial y} = (2x, 2y, 1).$$

As this vector is pointing upward (since it has has a positive z-component) it gives the desired orientation of D. Let $\tilde{D}$ be the disk of radius 2 in the (x, y)-plane, i.e. $\tilde{D}\{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 4\}$. Then

$$\begin{aligned}\iint_D \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_{\tilde{D}} \nabla \times \mathbf{F}(\mathbf{r}(x, y)) \cdot \big(\frac{\partial\mathbf{r}}{\partial x} \times \frac{\partial\mathbf{r}}{\partial y}\big)\,\mathrm{d}x\mathrm{d}y \\ &= \iint_{\tilde{D}} (-(3 + 2y)\,\mathbf{j} + 2z\,\mathbf{k}) \cdot (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})\,\mathrm{d}x\mathrm{d}y \\ &= \iint (-6y - 4y^2 + 10 - 2x^2)\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Substituting polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ this integral becomes

$$\begin{aligned}&\int_0^{2\pi}\int_0^2 (-6r\cos\theta - 6r^2\cos^2\theta + 10 - 2r^2\sin^2\theta)r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi} -2r^3\cos\theta - \frac{3}{2}r^4\cos^2\theta + 5r^2 - \frac{1}{2}r^4\sin^2\theta\Big|_{r=0}^{r=2}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} (-16\cos\theta - 24\cos^2\theta + 20 - 8\sin^2\theta)\,\mathrm{d}\theta \\ &= -16\sin\theta - 12\cos\theta\sin\theta - 12\theta + 20\theta + 4\cos\theta\sin\theta - 4\theta\Big|_{\theta=0}^{\theta=2\pi} \\ &= 8\pi\end{aligned}$$

which agrees with the result above.

6. Since the equation relates a volume integral to a surface integral of a vector field the statement is reminiscent of Gauß' Theorem. To show that the statement indeed follows from Gauß' Theorem we need to show that the divergence of the vector field in the surface integral is equal to the integrand of the volume integrals, i.e.

$$\nabla \cdot \frac{1}{r^2}\mathbf{r} = \frac{\partial}{\partial x}\frac{x}{r^2} + \frac{\partial}{\partial y}\frac{y}{r^2} + \frac{\partial}{\partial z}\frac{z}{r^2} = \frac{1}{r^2}.$$

We have

$$\frac{\partial}{\partial x}\frac{x}{r^2} = \frac{\partial}{\partial x}\frac{x}{x^2+y^2+z^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2},$$

where we used the quotient rule in the last equality. Similarly

$$\frac{\partial}{\partial y}\frac{y}{r^2} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2},$$

and

$$\frac{\partial}{\partial z}\frac{z}{r^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2},$$

Summing up these terms gives the desired result.